Homework 6, due Friday, July 15th

In this homework you will learn about linear dependence and linear independence.

In order to determine whether functions are linearly dependent or linearly independent we have to first understand **linear combinations**.

A **linear combination** of two functions, $f(x)$ and $g(x)$ is a sum $Af(x) + Bg(x)$ where A, B are real numbers. For example $3f(x) - 2g(x)$ is a linear combination where we take 3 of f(x) and subtract 2 of g(x).

1) Deterimine which of the following are linear combinations of $\sin(x)$ and $\cos(x)$. If yes, say what A and B are. If no, say why not.

a) $5\sin(x) + 3\cos(x)$

b) $\pi\sin(x) - e\cos(x)$

c) $5x^2\sin(x) + 2\cos(x)$

d) $-\sin(x) + \cos(x)$

e) $3\sin(x) - \sin(x)\cos(x)$

f) $17\sin(x) + \dfrac{1}{\cos(x)}$

Of course there is no reason that linear combinations have to stop at two functions. We can have for example $3x^2 + 2x - 5\sin(x) + \pi e^x$ which is a linear combination of the four functions $x^2, x, \sin(x), e^x$.

2) Consider the functions: $2x, \sin(x), \cos(x), 6x$. Find four constants A, B, C, D such that the linear combination of these four functions is 0. Here's one that you can't use since I'm giving it to you:

$$A = B = C = D = 0$$

because then we get $0 \cdot 2x + 0 \cdot \sin(x) + 0 \cdot \cos(x) + 0 \cdot 6x = 0 + 0 + 0 + 0 = 0$.

Since we are able to find A, B, C, D (not all being zero at once) which make a linear combination equal to zero we say the functions are **linearly dependent**. When you are unable to find such constants we say that functions are **linearly independent**.

Try to find two constants A, B (nonzero) such that the linear combination $Ax + Bx^2 = 0$. **(Hint: You can't)**. Let's try to show why we can't find the right nonzero A and B.

3) Suppose that there were such an A and B (not both zero) which made $Ax + Bx^2 = 0$ for any x.

a) What's the relationship between A and B if $x = 1$?

b) What's the relationship between A and B if $x = -1$?

c) What's the **ONLY** number that satisfies the two above properties? What does this tell you A and B have to be?

d) Are the functions x and x^2 linearly independent or linearly dependent?

Theorem: An nth-order linear homogeneous differential equation with constant coefficients always has n linearly independent solutions. The general solution to such an equation is then a linear combination of these solutions.
Use this to do the following problems.

(Hint: You may want to use the Wronskian as a fast way to see when things are linearly independent.)

4) Consider the second order linear homogeneous differential equation:

$$y'' - 2y' + y = 0$$

Two solutions are e^x and $5e^x$. Show that $y = c_1e^x + c_2 5e^x$ is not the general solution.

5) Consider the second order linear homogeneous differential equation:

$$y'' - 2y' + y = 0$$

Two solutions are e^x and xe^x. Show that $y = c_1e^x + c_2xe^x$ is the general solution.

6) Consider the third order linear homogeneous differential equation:

$$y''' - 6y'' + 11y' - 6y = 0$$

Two solutions are e^x and e^{2x}. Why is $y = c_1e^x + c_2e^{2x}$ not the general solution?

7) Consider the third order linear homogeneous differential equation:

$$y''' = 0$$

Three solutions are $1, x, x^2$. Is the general solution $y = c_1x^2 + c_2x + c_3$? Why or why not?

Practice Problems - Do not turn these in

Doing the following problems will benefit you. Practice makes perfect and **math is not a spectator sport**.

P1) Find the Wronskians of the given sets of functions and use this information to determine whether the given sets are linearly independent.

a) $3x, 4x$

b) x^3, x^2

c) $5, x^6$

d) e^x, e^{-x}

e) e^2x, e^{3x}

f) $\sin(x), \cos(x)$

g) $\sin(2x), \sin(-3x)$

h) e^2x, e^{3x}, e^{-x}

P2) For parts a)-h), try to show directly that they are linearly dependent or linearly independent without using the Wronskian.